SOME APPLICATIONS OF THE FUNK-HECKE THEOREM

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Theorem (Funk–Hecke)

Let $d \geq 2$, $k \in \mathbb{N}_0$ and $Y_k$ be a spherical harmonic of degree $k$. Then

$$\int_{S^{d-1}} F(\omega \cdot \theta) Y_k(\omega) \, d\omega = \lambda_k Y_k(\theta)$$

for any $\theta \in S^{d-1}$ and any function $F \in L^1([-1, 1], (1 - t^2)^{\frac{d-3}{2}})$. 

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for any $\theta \in S^{d-1}$ and any function $F \in L^1([-1, 1], (1 - t^2)^{\frac{d-3}{2}})$. Here, the constant $\lambda_k$ is given by

$$\lambda_k = |S^{d-2}| \int_{-1}^{1} F(t) P_{k,d}(t)(1 - t^2)^{\frac{d-3}{2}} \, dt$$

where $P_{k,d}$ is the Legendre polynomial of degree $k$ in $d$ dimensions.
Corollary

Let \( F \in L^1([-1, 1], (1 - t^2)^{\frac{d-3}{2}}) \) and let \( \lambda_k \) be as above in the Funk–Hecke theorem.
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Let $F \in L^1([-1, 1], (1 - t^2)^{\frac{d-3}{2}})$ and let $\lambda_k$ be as above in the Funk–Hecke theorem. Suppose that $G$ is a continuous function such that

$$\int_{S^{d-1}} F(\omega \cdot \theta) G(\theta) \, d\theta = 0$$

for each $\omega \in S^{d-1}$.

(1) $\lambda_k \neq 0$ for all $k \in \mathbb{N}_0 \Rightarrow G = 0$.

(2) $\lambda_k \neq 0$ for all odd $k \in \mathbb{N}_0 \Rightarrow G$ is even.

(3) $\lambda_k \neq 0$ for all even $k \in \mathbb{N}_0 \Rightarrow G$ is odd.
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Funk–Hecke theorem

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\Rightarrow \int_{(S^{d-1})^2} F(\omega \cdot \theta) Y_k(\omega) G(\theta) \, d\theta \, d\omega = \lambda_k \int_{S^{d-1}} Y_k(\theta) G(\theta) \, d\theta.
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For \((F, G)\) as in the corollary, we obtain

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Completeness of spherical harmonics in \(L^2(\mathbb{S}^{d-1})\) immediately implies (1).
Additionally, \(Y_k\) has same parity as \(k\). This yields (2) and (3).
A number of geometric quantities (such as volumes, $d - 1$-dimensional section areas,...) may be expressed in terms of

\[
\int_{\omega \cdot \theta \geq 0} G(\theta)(\theta \cdot \omega) \ell \, d\theta \quad \text{or} \quad \int_{\omega \cdot \theta = 0} G(\theta) \, d\theta
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in which case the corollary is often applicable.
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The Busemann–Petty problem

Busemann–Petty (1956): If $K_1$ and $K_2$ are origin-symmetric convex bodies in $\mathbb{R}^d$ such that
$$\text{vol}_{d-1}(K_1 \cap H) \leq \text{vol}_{d-1}(K_2 \cap H)$$
for all hyperplanes $H$ containing the origin, is it true that
$$\text{vol}_d(K_1) \leq \text{vol}_d(K_2)?$$

No for $d \geq 5$: Larman–Rogers ($d \geq 12$), Ball ($d \geq 10$), Giannopoulos, Bourgain ($d \geq 7$), Gardner, Papadimitrakis ($d \geq 5$).

Yes for $d \leq 4$: Gardner ($d = 3$), Zhang ($d = 4$).

Short proof of $d = 4$ by Koldobsky using Funk–Hecke theorem.
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Applications in analysis and PDE

Suppose $T$ is the integral operator on $L^2(S^{d-1})$ given by

$$Tg(\theta) = \int_{S^{d-1}} g(\omega) K(\omega \cdot \theta) \, d\omega.$$
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Expanding $g = \sum_{k \in \mathbb{N}_0} Y_k$,

$$Tg(\theta) = \sum_{k \in \mathbb{N}_0} \lambda_k Y_k(\theta)$$

where

$$\lambda_k = |S^{d-2}| \int_{-1}^{1} K(t)P_{k,d}(t)(1 - t^2)^{\frac{d-3}{2}} dt.$$
Lieb’s sharp HLS inequality: On $\mathbb{R}^d$, with $\lambda \in (0, d)$:

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \right| \leq \pi^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(d - \frac{\lambda}{2}\right)} \left( \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{1-\lambda}{d}} \|f\|_p \|g\|_p$$

where $p := \frac{2d}{2d-\lambda}$. 

Frank–Lieb: recent rearrangement-free proof using Funk–Hecke theorem. 

They proved with the same technique the analogous sharp HLS inequality on the Heisenberg group $H^n$. 

For $H^n$, previously sharp form only known in a special case of $\lambda$ (Jerison–Lee). 

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Beckner: solved conjecture of Stein on contraction properties of Poisson semigroup on $S^d$ using Lieb’s sharp HLS and Funk–Hecke theorem.
Sharp weighted Fourier extension/Kato-smoothing

For \((x, t) \in \mathbb{R}^d \times \mathbb{R}\), consider

\[Sf(x, t) = |x| - \tau \int_{\mathbb{R}^d} e^{i(x \cdot \xi + \frac{1}{2} t |\xi|^2)}|\xi|^{1-\tau} f(\xi) \, d\xi\]

where \(\tau \in (1, d)\).

Interpretation as Fourier extension operator associated with the paraboloid

\[
\{ (\xi, \frac{1}{2} |\xi|^2) : \xi \in \mathbb{R}^d \} \subset \mathbb{R}^d + 1,
\]

or as a solution operator through

\[\hat{S}f(x, t) = (2\pi)^d \frac{1}{x| |x|^2 - \tau (\frac{1}{2} \Delta)^{1/2} \tau u(x, t)}\]

where \(i \partial_t u + \frac{1}{2} \Delta u = 0\) with initial data \(f\).
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S\hat{f}(x, t) = (2\pi)^d |x|^{-\frac{\tau}{2}} (-\Delta)^{\frac{1}{2} - \frac{\tau}{4}} u(x, t)
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where \(i\partial_t u + \frac{1}{2} \Delta u = 0\) with initial data \(f\).
Recall

\[ Sf(x, t) = |x|^{-\frac{\tau}{2}} \int_{\mathbb{R}^d} e^{i(x \cdot \xi + \frac{1}{2} t|\xi|^2)} |\xi|^{1-\frac{\tau}{2}} f(\xi) \, d\xi. \]
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**Theorem (B–Sugimoto)**

For each $k \in \mathbb{N}_0$,

$$S^* Sf = \lambda_k f$$

where

$$f(\eta) = Y_k(\frac{\eta}{|\eta|}) f_0(|\eta|)|\eta|^{-\frac{d-1}{2}},$$

$Y_k$ is any spherical harmonic of degree $k$, $f_0 \in L^2(0, \infty)$, and

$$\lambda_k = (2\pi)^{d+1} 2^{1-\tau} \frac{\Gamma(\tau - 1) \Gamma(k + \frac{d-\tau}{2})}{\Gamma(\frac{\tau}{2})^2 \Gamma(k + \frac{d+\tau}{2} - 1)}.$$
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Moreover, \((\lambda_k)_{k \in \mathbb{N}_0}\) is a strictly decreasing sequence.
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Hence, \( \|S\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})} = \sqrt{\lambda_0} \) which is attained if and only if \( f \) is radially symmetric.
Relabelling $s = \frac{\tau}{2} - 1$ ($\tau \in (1, d) \iff s \in (-\frac{1}{2}, \frac{d}{2} - 1)$).
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**Corollary (Simon, Watanabe)**

Let $d \geq 2$ and $s \in (-\frac{1}{2}, \frac{d}{2} - 1)$. If $i \partial_t u + \frac{1}{2} \Delta u = 0$ then

$$\int \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{dx dt}{|x|^{2(s+1)}} \leq C(d, s)\|u(0)\|^2_{\dot{H}^s(\mathbb{R}^d)},$$

where

$$C(d, s) = \pi 2^{-2s} \frac{\Gamma(2s + 1)\Gamma(\frac{d}{2} - s - 1)}{\Gamma(s + 1)^2\Gamma(\frac{d}{2} + s)}.$$

The constant is sharp and equality holds if and only if $u(0) \in \dot{H}^s(\mathbb{R}^d)$ is radially symmetric.
A calculation shows that

\[ S^* S f(\eta) = C(d, \tau) \int_{S^{d-1}} \frac{f(|\eta|\omega)}{|\omega - \eta'|^{d-\tau}} \, d\omega \]

for some explicitly computable constant \( C(d, \tau) \). Here, \( \eta' = |\eta| \).
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Applied to

\[ f(\eta) = Y_k(\frac{\eta}{|\eta|}) f_0(|\eta|) |\eta|^{-\frac{d-1}{2}} \]

the angular and radial variables split, and the result follows from the Funk–Hecke theorem.
The \( \frac{1}{2} \)-derivative endpoint: localisation

Recall that for \( d \geq 2 \) and \( s \in (-\frac{1}{2}, \frac{d}{2} - 1) \),

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(x, t)|^2 \frac{\mathrm{d}x\mathrm{d}t}{|x|^{2(s+1)}} \leq C(d, s)\|u(0)\|_{H^s(\mathbb{R}^d)}^2,
\]

whenever \( i\partial_t u + \frac{1}{2} \Delta u = 0 \), and this fails for \( s = -\frac{1}{2} \).
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whenever $i \partial_t u + \frac{1}{2} \Delta u = 0$, and this fails for $s = -\frac{1}{2}$.

One recovers a full half-derivative gain by spatial localising

$$\sup_{R > 0} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| \leq R} |\nabla u(x, t)|^2 \, dx dt \leq C \|u(0)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}^2.$$

Proved separately by Constantin–Saut, Sjölin and Vega.
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$$\sup_{R>0} \frac{1}{R} \int_{\mathbb{R}} \int_{|x| \leq R} |\nabla u(x, t)|^2 dxdt \geq c\|u(0)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}.$$
The $\frac{1}{2}$-derivative endpoint: angular regularity

Theorem ($\leq$ Hoshiro, Sugimoto; $\geq$ Fang–Wang)

Let $d \geq 2$ and $s \in (-\frac{1}{2}, \frac{d}{2} - 1)$. Then there exist constants

$$0 < c(d, s) \leq C(d, s) < \infty$$

such that whenever $i\partial_t u + \frac{1}{2} \Delta u = 0$ we have
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$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |(1 - \Lambda)^{\frac{1+2s}{4}} u(x, t)|^2 \frac{dx dt}{|x|^{2(1+s)}} \leq C(d, s) \|u(0)\|_{H^s(\mathbb{R}^d)}^2$$

and

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Here, $\Lambda$ is the Laplace–Beltrami operator on $S^{d-1}$. 
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$$b = \inf_{\ell \in \mathbb{N}_0} \beta_\ell, \quad B = \sup_{\ell \in \mathbb{N}_0} \beta_\ell,$$

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Write \( \mathcal{H}_k \) for the space of all linear combinations of functions

\[
\eta \mapsto Y_k(\frac{\eta}{|\eta|}) f_0(|\eta|)|\eta|^{-\frac{d-1}{2}}
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where \( Y_k \) is a spherical harmonic of order \( k \) and \( f_0 \in L^2(0, \infty) \).
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If \( i \partial_t u + \frac{1}{2} \Delta u = 0 \) then

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\]

and the constants are sharp. Also, \( u(0) \) is an extremiser for the lower bound if and only if \( D_s u(0) \in \bigoplus_{k \in K} H^k \), and an extremiser for the upper bound if and only if \( D_s u(0) \in \bigoplus_{k \in K} H^k \).

Hoshiro, Sugimoto and Fang–Wang equivalences correspond to \( \theta(\rho) = (1 + \rho)^{1+2s/4} \).

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$$k = \{k \in \mathbb{N}_0 : b = \beta_k\}, \quad K = \{k \in \mathbb{N}_0 : B = \beta_k\}.$$ 

For $d \geq 5$, define $\tau_*, \tau^* \in (1, d)$, $\tau_* \leq \tau^*$, by

$$d^{\frac{\tau_*-1}{2}} \left(\frac{d - \tau_*}{2}\right) = \frac{d + \tau_*}{2} - 1 \quad \text{and} \quad \Gamma\left(\frac{d-\tau^*}{2}\right) = \Gamma\left(\frac{d+\tau^*}{2} - 1\right).$$
For $d = 5$ and $\tau \in [\tau_*, 5)$, let $k(\tau) \in [0, \infty)$ satisfy

$$\frac{2k(\tau) + 5 - \tau}{2k(\tau) + 3 + \tau} \left( \frac{1 + (k(\tau) + 1)(k(\tau) + 4)}{1 + k(\tau)(k(\tau) + 3)} \right)^{\frac{\tau-1}{2}} = 1$$

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We know

\[
\frac{C_1}{(5 - \tau)^{1/4}} \leq k(\tau) \leq \frac{C_2}{(5 - \tau)^{1/2}}
\]

for some positive constants \( C_1 \) and \( C_2 \).
The case $\theta(\rho) = (1 + \rho) \frac{\tau - 1}{4}$, $\tau \in (1, d)$

**Theorem (B–Sugimoto)**

Let $d \geq 2$, $\tau \in (1, d)$ and $\theta(\rho) = (1 + \rho) \frac{\tau - 1}{4}$. Then
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<th>$\beta$</th>
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<tbody>
<tr>
<td>$d = 2, 3$</td>
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Sharp trace theorem on $S^{d-1}$

Let $Rf := f|_{S^{d-1}}$ and
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Let $Rf := f|_{S^{d-1}}$ and $T_\theta g := R\tilde{\theta}(-\Lambda)D^{-s}g$. Also

$$\lambda_k = 2^{1-2s} \frac{\Gamma(2s-1)\Gamma(k + \frac{d}{2} - s)}{\Gamma(s)^2\Gamma(k + \frac{d}{2} - 1 + s)} |\theta(k(k + d - 2))|^2.$$ 

and $K = \{ k \in \mathbb{N}_0 : \lambda_k = \sup_{\ell \in \mathbb{N}_0} \lambda_{\ell} \}$. 

Theorem (B–Machihara–Sugimoto)

Let $d \geq 2$ and $s \in (\frac{1}{2}, \frac{d}{2})$. Then

$$\|R\tilde{\theta}(-\Lambda)f\|_{L^2(S^{d-1})} \leq \sup_{k \in \mathbb{N}_0} \lambda_k \|f\|_{\dot{H}^s(R^d)}$$

where the constant is optimal and the space of extremisers is precisely the nonzero elements of $D^{-s}T^*\theta(\bigoplus_{k \in K} H_k)$.
Sharp trace theorem on $S^{d-1}$

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**Theorem (B–Machihara–Sugimoto)**

*Let $d \geq 2$ and $s \in \left(\frac{1}{2}, \frac{d}{2}\right)$. Then*

$$\| R\overline{\theta}(-\Lambda)f \|_{L^2(S^{d-1})}^2 \leq \sup_{k \in \mathbb{N}_0} \lambda_k \| f \|_{H^s(\mathbb{R}^d)}^2$$

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Let $Rf := f|_{\mathbb{S}^{d-1}}$ and $T_\theta g := R \bar{\theta}(-\Lambda)D^{-s}g$. Also

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**Theorem (B–Machihara–Sugimoto)**

*Let $d \geq 2$ and $s \in \left(\frac{1}{2}, \frac{d}{2}\right)$. Then*

$$\| R \bar{\theta}(-\Lambda)f \|^2_{L^2(\mathbb{S}^{d-1})} \leq \sup_{k \in \mathbb{N}_0} \lambda_k \| f \|^2_{H^s(\mathbb{R}^d)}$$

*where the constant is optimal and the space of extremisers is precisely the nonzero elements of $D^{-s} T_\theta^*(\bigoplus_{k \in K} \mathcal{H}_k)$.*

Trace theorems with angular regularity considered by Fang–Wang.
When $\theta = 1$ we have

$$T_1 T_1^* G(\omega) = 2^{-2s} \pi^{-d/2} \frac{\Gamma(d/2 - s)}{\Gamma(s)} \int_{\mathbb{S}^{d-1}} \frac{G(\varphi)}{|\omega - \varphi|^{d-2s}} \, d\sigma(\varphi)$$

and we may apply the Funk–Hecke theorem.
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When $d = 3$, for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$,

$$|\cdot|^{2s-3} \ast d\sigma(x) = C(s) \frac{(|x| + 1)^{2s-1} - |x| - 1}{|x|}.$$
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For $(d, s) = (3, 1)$, extremisers for

$$\|f\|_{S^2} \leq \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

are precisely nonzero multiples of

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ \frac{1}{|x|} & \text{if } |x| > 1 \end{cases}.$$
Strichartz estimates for the wave equation

For $d \geq 2$ and $s \in \left[\frac{1}{2}, \frac{d}{2}\right)$,

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^{d+1})} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

for each $f \in \dot{H}^s(\mathbb{R}^d)$ and where

$$p = \frac{2(d+1)}{d - 2s}.$$
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The sharp constant and a full characterisation of extremisers is only known in some rather isolated cases.
It is known that, for all admissible \((d, s)\), an extremiser exists (Bulut, Fanelli–Vega–Viscigia, Ramos).
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Finding the exact shape of such extremisers appears to be a rather difficult problem; known when \((d, s) = (2, \frac{1}{2}), (3, \frac{1}{2})\) (Foschi) and \((d, s) = (5, 1)\) (B-Rogers).

In each of these cases, the initial datum \(f_{\star}\) such that

\[
\hat{f}_{\star}(\xi) = \frac{e^{-|\xi|}}{|\xi|}
\]

is extremal.
Theorem (B-Jeavons)

The one-sided wave propagator satisfies the estimate

\[ \| e^{it\sqrt{-\Delta}} f \|_{L^4(\mathbb{R}^5)} \leq \left( \frac{4}{15\pi^2} \right)^{\frac{1}{4}} \| f \|_{\dot{H}^\frac{3}{4} (\mathbb{R}^4)} \]

where the constant is sharp and is attained if and only if

\[ \hat{f}(\xi) = \frac{e^{a|\xi| + ib \cdot \xi + c}}{|\xi|}, \]

where \( a, c \in \mathbb{C} \) such that \( \text{Re}(a) < 0 \), and \( b \in \mathbb{R}^d \).
Theorem (B-Rogers)

Let $d \geq 3$. Then

$$\|e^{it\sqrt{-\Delta}}f\|_{L^4(\mathbb{R}^{d+1})}^4 \leq C(d) \int_{(\mathbb{R}^d)^2} |\hat{f}(y_1)|^2 |\hat{f}(y_2)|^2 |y_1|^\frac{d-1}{2} |y_2|^\frac{d-1}{2} (1 - y'_1 \cdot y'_2)^\frac{d-3}{2} dy_1 dy_2$$

holds with sharp constant

$$C(d) = 2^{-\frac{d-1}{2}} (2\pi)^{-3d+1} |\mathbb{S}^{d-1}|$$
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which is attained if and only if

$$\hat{f}(\xi) = \frac{e^{a|\xi|+b \cdot \xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$ with $|\text{Re}(b)| < -\text{Re}(a)$.
Polar coordinates gives

\[ \| e^{it\sqrt{-\Delta} f} \|_{L^4(\mathbb{R}^{d+1})}^4 \leq \frac{C(d)}{2^{d-3}} \int_{(S^{d-1})^2} g(\omega_1)g(\omega_2)|\omega_1 - \omega_2|^{d-3} d\omega_1 d\omega_2 \]

where

\[ g(\omega) = \int_0^\infty |\hat{f}(r\omega)|^2 r^{\frac{3(d-1)}{2}} dr. \]
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where

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Note that

\[\int_{\mathbb{S}^{d-1}} g = (2\pi)^d \|f\|_{H^{\frac{d-1}{4}}}^2.\]
Let
\[ H_\rho(g) = \int_{(S^{d-1})^2} g(\omega_1)g(\omega_2)|\omega_1 - \omega_2|^{-\rho} \, d\omega_1 d\omega_2 \]
Let

\[ H_\rho(g) = \int_{(S^{d-1})^2} g(\omega_1)\overline{g(\omega_2)}|\omega_1 - \omega_2|^{-\rho} \, d\omega_1 d\omega_2 \]

and \( \mu_g = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} g \).
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and \( \mu_g = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} g. \)

**Theorem (B-Jeavons)**

Let \(-2 < \rho < 0\), and let \( g \in L^1(S^{d-1}). \) Then
\[
H_{\rho}(g) \leq H_{\rho}(\mu_g 1) = 2^{d-2-\rho} B\left(\frac{d-1-\rho}{2}, \frac{d-1}{2}\right) \frac{|S^{d-2}|}{|S^{d-1}|} \left| \int_{S^{d-1}} g \right|^2
\]
and equality holds if and only if \( g \) is constant.
Expanding $g = \sum_{k \in \mathbb{N}_0} Y_k$, the Funk–Hecke theorem gives

$$H_\rho(g) = 2^{-\frac{\rho}{2}} \sum_{k \geq 0} I_k(d, \rho) \int_{S^{d-1}} |Y_k(\omega)|^2 \, d\omega$$

where

$$I_k(d, \rho) = |S^{d-2}| \int_{-1}^{1} (1 - t)^{-\frac{\rho}{2}} P_{k,d}(t)(1 - t^2)^{\frac{d-3}{2}} \, dt.$$ 

**Lemma**

If $-2 < \rho < 0$ then

$$I_0(d, \rho) = |S^{d-2}| 2^{d-2-\frac{\rho}{2}} B\left(\frac{d-1-\rho}{2}, \frac{d-1}{2}\right) > 0$$

and $I_k(d, \rho) < 0$ for all $k \geq 1$. 
Hence

\[ H_{\rho}(g) \leq 2^{-\frac{\rho}{2}} I_0(d, \rho) \int_{S^{d-1}} |Y_0|^2 \, d\omega = H_{\rho}(\mu_g 1). \]
Hence

\[ H_\rho(g) \leq 2^{-\frac{\rho}{2}} I_0(d, \rho) \int_{S^{d-1}} |Y_0|^2 \, d\omega = H_\rho(\mu_g \mathbf{1}). \]

The lemma fails for \( \rho < -2 \) because \( I_2(d, \rho) > 0 \).
Hence

\[ H_\rho(g) \leq 2^{-\frac{\rho}{2}} I_0(d, \rho) \int_{S^{d-1}} |Y_0|^2 \, d\omega = H_\rho(\mu_g \mathbf{1}). \]

The lemma fails for \( \rho < -2 \) because \( I_2(d, \rho) > 0 \).

Applying it with \((d, \rho) = (4, -1)\) gives

\[
\| e^{it\sqrt{-\Delta}} f \|_{L^4(\mathbb{R}^{4+1})}^4 \leq \frac{C(4)}{\sqrt{2}} \int_{(S^3)^2} g(\omega_1)g(\omega_2)|\omega_1 - \omega_2| \, d\omega_1 d\omega_2
\]

\[
\leq \frac{C(4)}{\sqrt{2}} H_{-1}(1)|\mu_g|^2
\]

\[
= \frac{4}{15\pi^2} \| f \|^4_{\dot{H}^1(\mathbb{R}^4)}.
\]
For higher dimensions, we want to use the case \((d, \rho) = (d, 3 - d)\). But \(3 - d < -2\) for \(d \geq 6\).
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The case \((d, \rho) = (3, -1)\) was used by Foschi to prove that constant functions are extremisers in the Stein–Tomas inequality

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\| \hat{g}d\sigma \|_{L^4(\mathbb{R}^3)} \leq C \| g \|_{L^2(d\sigma)}.
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This was partially extended to higher dimensions by Carneiro–Oliveira e Silva, with a similar obstruction preventing a generalisation to all dimensions.
Thanks for listening....!