Unions of lines in (vector spaces over) finite fields

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Suppose $L$ is a collection of lines in $\mathbb{R}^d$ such that for every $\xi \in S^{d-1}$ there is an $l \in L$ with $l$ parallel to $\xi$.

Then $\text{hdim}(\bigcup_{l \in L} l) = d$

Or slightly stronger:

Suppose $L$ is a collection of lines in distinct directions.

Then $\text{dim}(\bigcup_{l \in L} l) \geq \text{dim}(L) + 1$
Wolff’s estimate

Suppose $L$ is a collection of lines in $\mathbb{R}^d$ such that for every $\xi \in S^{d-1}$ there is an $l \in L$ with $l$ parallel to $\xi$.

Then $\text{hdim}(\bigcup_{l \in L} l) \geq \frac{d+2}{2}$

Also proved stronger:

Suppose $L$ is a collection of lines satisfying the “Wolff axiom”

Then $\text{dim}(\bigcup_{l \in L} l) \geq (\text{dim}(L) + 3)/2$

for $\text{dim}(L) \geq 1$
Wolff axioms

Weak Wolff axiom: For every 2-plane $P$, \( \dim(\{l \in L : l \subset P\}) \leq 1 \)

Strong Wolff axiom: For every $l_0 \subset P$ and $0 < \delta < R$: there are at most $\frac{R}{\delta}$ $\delta$-separated lines from $L$ contained in $P$ within distance $R$ from $l_0$. 
Q: (D. Oberlin) Can you give a nontrivial lower bound for \( \dim(\bigcup_{l \in L} l) \) in terms of \( \dim(L) \) \textit{with no assumptions placed on } L?  

\textit{Bush argument:}  \( \text{hdim}(\bigcup_{l \in L} l) \geq \text{hdim}(L)/2 + 1 \)

\textit{Fourier analysis:}  \( \text{hdim}(\bigcup_{l \in L} l) \geq \text{hdim}(L) + 1, \ 0 \leq \text{hdim}(L) \leq 1 \)
Theorem: (D. Oberlin) Suppose $L$ is a collection of lines in $\mathbb{R}^3$. If $0 \leq \beta \leq 1$ and $\text{hdim}(L) \geq 2 + \beta$ then $\text{hdim}(\bigcup_{l \in L} l) \geq 2 + \beta$.

Proof: Fourier analysis

![Graph showing the relationship between dim(L) and dim(Union)]
Conjecture: (D. Oberlin) Suppose $L$ is a collection of lines in $\mathbb{R}^n$, that $1 \leq d < n$ is an integer, and that $0 \leq \beta \leq 1$.

If $\operatorname{hdim}(L) \geq 2(d - 1) + \beta$ then $\operatorname{hdim}(\bigcup_{l \in L} l) \geq d + \beta$
Finite fields

$F$ – finite field

$F^n$ in place of $R^n$ (model case)

$E \subset F^n$ has “dimension” $\geq \alpha$ if $|E| \gtrsim |F|^\alpha$

$L$ has “dimension” $\geq \alpha$ if $|L| \gtrsim |F|^\alpha$

Want estimates uniform over all $F$. Then $|F| \to \infty$, is analogous to $\delta \to 0$.

Pose the same conjecture in $F^n$
**Theorem:** (R. Oberlin) If $1 \leq d < n$ is an integer and $L$ is a collection of lines in $F^n$ with $|L| \geq \log(|F|)^d |F|^{2(d-1)+\beta}$ then

$$|\bigcup_{l \in L} l| \geq c_d |F|^{d+\beta}.$$ 

**Proof:** Geometric combinatorics (hairbrush style)

**Corollary:** For even integers $n$, Wolff’s $\frac{n+2}{2}$ “Kakeya bound” in finite fields holds even for sets of lines that do not obey the Wolff axiom.
Possible extensions

$k$-planes instead of lines?

**Conjecture:** Suppose $P$ is a collection of $k$-planes, and $0 \leq \beta \leq 1$. If $\dim(P) \geq (k + 1)(d - k) + \beta$ then $\dim(\bigcup_{p \in P} p) \geq d + \beta$. 

![Graph showing dim(Union) vs. dim(P) for k=2]
Hyperplanes instead of points?

For $k$-planes $p$, let $Q_p$ be the set of $(k - 1)$-planes contained in $p$.

**Conjecture:** Suppose $P$ is a collection of $k$-planes, and $0 \leq \beta \leq k$. If $\dim(P) \geq (k + 1)(d - k) + \beta$ then

$$
\dim\left(\bigcup_{p \in P} Q_p\right) \geq ((k - 1) + 1)(d - (k - 1)) + \beta.
$$
$k'$-planes instead of hyperplanes?

For $k$-planes $p$ and $0 \leq k' < k$, let $Q_{p,k'}$ be the set of $k'$-planes contained in $p$.

**Conjecture:** Suppose $P$ is a collection of $k$-planes, $k' < k$, and $0 \leq \beta \leq k' + 1$. If $\dim(P) \geq (k + 1)(d - k) + \beta$ then

$$\dim\left( \bigcup_{p \in P} Q_{p,k'} \right) \geq (k' + 1)(d - k') + \beta.$$
Proof, $k = 1$

$d = 1 : (\text{Córdoba})$

$|L| = |F|$

Want $|\bigcup_{l \in L} l| \gtrsim |F|^2$

**Fact**: Each line $l \in L$ intersects each line $l' \in L$ at most once

So $\frac{1}{|I|} \sum_{x \in I} |\{l' \in L : x \in l'\}| \leq \frac{2|F|}{|F|} \leq 2$

So “average point on each line” incident to $\leq 2$ lines
$d = 2$

$|L| = |F|^3$, want $|\bigcup_{l \in L} l| \gtrsim |F|^3$

Case 1: Wolff axiom holds
“For each 2-plane $P$ at most $|F|$ lines from $L$ are contained in $P$”

If most points on each line are contained in $\leq |F|$ other lines then done.

Otherwise some line intersects $\gtrsim |F|^2$ other lines.
Foliate into 2-planes. Apply Córdoba to the lines lying in each 2-plane. Add up the points, and done.
Case 2: Wolff axiom does not hold.

Then there is a 2-plane $P_0$ so that more than $|F|$ lines from $L$ are contained in $P_0$.

Applying Córdoba, essentially $P_0 \subset \bigcup_{l \in L} l$

Set $L_0 = L \setminus \{l : l \subset P_0\}$

If $L_0$ satisfies Wolff axiom, Proceed via case 1. Otherwise pick $P_1$ containing at least $|F|$ lines from $L_0$.

Continue until either most of the lines are in $P_0 \cup \cdots \cup P_N$ or the Wolff axiom is satisfied for $L_N$.

If latter, done.
At most $|F|^2$ lines in each $P_j$ so at least $|F|$ of the $P_j$.

Can union that many up via Córdoba, giving $|\bigcup_j P_j| \gtrapprox |F|^3$ and done.
$d = 3$

Have $|L| \geq |F|^5$, want $|\bigcup_{l \in L} l| \gtrsim |F|^4$

Proceed as before. If Wolff satisfied, then done.

Otherwise obtain saturated planes $P_1, \ldots, P_N$ with $N \gtrsim |F|^3$.

Too many for Córdoba!
(A) Wolff axiom for 2-planes: “At most $|F|$ 2-planes $P_j$ in each 3-plane $Q$”

(stronger than Alvarez, Mitsis, & Bueti condition)

If not satisfied, then gather $P_j$ into saturated 3-planes $Q_k$.

At most $|F|^4$ lines in each $Q_k$ so at most $|F|^2$ of the $P_j$ in each $Q_k$ so at least $|F|$ of the $Q_k$.

Apply Córdoba to $Q_k$ and done.
If Wolff satisfied, then build a hairbrush of 2-planes.

Then foliate into 3-planes, and use Córdoba in each.
Works perfectly if 2-planes only intersect in lines (not points).

Which is all you have to worry about for $d = 3$.

But for $d \geq 4$ point intersections are a problem.
General $d$

Suppose proven for $d' < d$.

Choose “maximal” $m$ so that the lines saturate (according to $d' < d$ theorem) $m$-planes.

Thus, an appropriate Wolff condition satisfied for $m'$ planes, $m' > m$.

If most points not contained in too many $m$-planes, done.

Otherwise most points in a given $m$-plane $P$ are contained in a lot of $m$-planes $P'$, and for each of those $P'$ the point is contained in a lot of lines.
Use to build hairbrush of lines emanating from $m$-plane

Foliate into $(m + 1)$-planes and done.